Symmetries in algebraic geometry and motives

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16 May 2023

Introduction

Symmetries in mathematics

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From this perspective, Algebraic Geometry is no different.

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Observe the analogy with rational points on curves!

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The talk will be (mostly) very very elementary...
I'll concentrate mainly on the philosophical aspects...
Statements will be rather imprecise at the beginning and could be misinterpreted... (Please interrupt me if needed!)
I'll try to give some precise statements towards the end...

Extending the concept of symmetry

A first try

As automorphisms are rare in Algebraic Geometry, it is natural to look for substitutes.

For instance, one may consider the endomorphisms of X. These are maps $f : X \to X$ preserving all the structures but with the notable difference that f may contract X onto a *strict* sub-variety of X.

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Unfortunately, in Algebraic Geometry endomorphisms are as rare as automorphisms!

The non-identity endomorphisms of a general algebraic variety tend to have a constant image, i.e., contract the whole variety to a point.

Outgoing and ingoing quasi-symmetries

As endomorphisms are rare in Algebraic Geometry, it is often useful, given an algebraic variety X, to "look outside" X and compare it with other algebraic varieties:

> It is useful to consider maps from X (i.e., $X \to P$) and maps to X (i.e., $Q \to X$).

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> To study X, it can be useful to consider it as *an object of the category* of algebraic varieties (or maybe a well chosen subcategory of these).

The lack of symmetries is maybe a good explanation for the "unreasonable omnipresence" of Category Theory in some parts of Algebraic Geometry!

Maps from a fixed variety (or outgoing quasi-symmetries)

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For instance, if P is the zero set in $\mathbb{C}^N$ of a system of polynomial equations

$$(\mathcal{S}) : \begin{cases} p_1(t_1, \dots, t_N) &= 0, \\ &\vdots \\ p_m(t_1, \dots, t_N) &= 0, \end{cases}$$

then f is a rule which associates to every $x \in X$ a solution

$$f(x) = (f_1(x), \dots, f_N(x))$$

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of the system $(\mathcal{S})$.
An important requirement is that the f_i's are "algebraic functions" of the variable $x \in X$.

Maps from a fixed variety (continued)

From the previous slide, there is an equivalence between the following data:

- A (nontrivial) map $f : X \to P$;
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Observation (very imprecise)

In practice, this condition forces $(\mathcal{S})$ to be a sub-system of the system of equations defining X (e.g., the empty system) or a system obtained from those equations by standard universal operations.

Maps from a fixed variety (continued)

Recapitulation

1. The existence of an interesting map $f : X \to P$ may have strong implications on the geometry of X. This is especially the case when P itself has an interesting geometry.
2. Unfortunately, maps from a fixed algebraic variety X are rarely interesting. For a general X, f is an immersion or P tends to have a boring geometry (e.g., $P = \mathbb{P}^n$).

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Remark

Although maps from X to simple varieties such as $\mathbb{P}^N$ are relatively uninteresting, they play a prominent role in Algebraic Geometry especially in studying the global geometry of projective varieties.
But this is a different story...
Here we are interested in the local geometry of algebraic varieties...

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Observation

A map $g : Q \to X$ corresponds to adding new variables and new equations to the system of equations defining X.

Indeed, one gets $g : Q \to X$ by cutting a sub-variety Q of the cartesian product $X \times \mathbb{C}^N$ using a system of equations

$$(\mathcal{T}) : \begin{cases} q_1(x, t_1, \dots, t_N) & = & 0, \\ & \vdots & \\ q_m(x, t_1, \dots, t_N) & = & 0. \end{cases}$$

Here the q_i's are polynomials in the variables t_j's depending algebraically on $x \in X$.

Maps to a fixed variety (continued)

Example

Assume that $m = N = 1$ and that

$$q(x,t) = t^d + a_{d-1}(x) \cdot t^{d-1} + \cdots + a_0(x)$$

is a monic polynomial in t with coefficients in the ring of algebraic functions on X. In this case, $f : Q \to X$ is a *finite cover* of X.

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Indeed, we have

$$Q = \{(x,t) \in X \times \mathbb{C} \mid q(x,t) = 0\}.$$

Therefore, for $x_0 \in X$, $f^{-1}(x_0)$ can be identified with the set of d roots (counted with multiplicities) of the equation

$$t^d + a_{d-1}(x_0) \cdot t^{d-1} + \cdots + a_0(x_0) = 0.$$

In particular, if $q(x,t)$ is irreducible, f is generically a $d\!-\!\text{to}\!-\!1$ map.

The formalism of descent

Recapitulation and some natural questions

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Thus, it is useful to develop techniques for studying X by systematically exploiting maps of the form $g : Q \to X$.

Questions/Problems

- Classify sorts of maps $g : Q \to X$.
- Find methods to descend/transfer informations from Q to X, in good situations.

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- Étale surjective maps are very well suited for descending informations from the source (i.e., Q) to the target (i.e., X).
- The étale topology – also invented by Grothendieck – is a powerful tool to study/perform descent along étale maps.

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$$Q_n = \overbrace{Q \times_X \cdots \times_X Q}^{n+1 \text{ times}}.$$

Thus Q_n is the set of $(n+1)$-tuples $(y_0, \ldots, y_n)$ of points in Q such that $g(y_0) = \cdots = g(y_n)$.

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one for every (increasing) function $r : [\![0, m]\!] \to [\![0, n]\!]$.

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This geometric-combinatorial data $Q_\bullet$ is called a simplicial variety.

Descending along étale maps (continued)

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The Cech complex is actually a particular example of what is called an étale hyper-cover. This is a *simplicial variety* $Y_\bullet$ together with an augmentation $Y_\bullet \to X$ such that:

1. $Y_0 \to X$ is an étale surjective map;
2. $Y_1 \to Y_0 \times_X Y_0$ is an étale surjective map;
3. $Y_2 \to (Y_1 \times_{Y_0} Y_1) \times_{(Y_0 \times_X Y_0)} Y_1$ is an étale surjective map;
4. etc.

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General Principle

Given an étale hyper-cover $Y_\bullet \to X$, there is an equivalence between:

1. Properties/invariants of X;
2. Properties/invariants of the Y_i's which are compatible with the simplicial structure.

Descending along étale maps (continued)

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Remark 2

In practice, the Y_i's are much more complicated than X itself. (This is peculiar to algebraic geometry; in differential geometry there are hyper-covers given by disjoint unions of balls in each simplicial degree.)

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Remark 2

In practice, the Y_i's are much more complicated than X itself. (This is peculiar to algebraic geometry; in differential geometry there are hyper-covers given by disjoint unions of balls in each simplicial degree.)
Therefore, the previous principle is only theoretical...
In practice, it is unreasonable to expect this principle to be useful...
But wait for the next slides...

Motivic quasi-symmetries

É-morphisms: the idea

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2. Given an étale hyper-cover $Y_\bullet \to X$, we may look for interesting maps $Y_\bullet \to W$.
3. Given such a map $Y_\bullet \to W$, we may transfer (contravariant) informations from W to $Y_\bullet$ and then descend these to X.
   (If $W = X$, this could serve as an endomorphism of X.)

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   (If $W = X$, this could serve as an endomorphism of X.)
4. These Zigzags from X to W are what we will call today the é-morphisms.

É-morphisms: more precisely

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These invariants are contravariant in X, i.e., given a map $h : X \to W$, one has a homomorphism $h^* : H^*(W) \to H^*(X)$. We will need an obvious extension of this functoriality.

Observation

Given a finite family of maps $h_1, \ldots, h_m : X \to W$, and integers $a_1, \ldots, a_m \in \mathbb{Z}$, there is a map

$$\sum_{i=1}^{m} a_i h_i^* : H^*(W) \to H^*(X).$$

É-morphisms: more precisely (continued)

Definition

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2. a (formal) linear combination $\sum_{i=1}^{m} a_i \cdot h_i$ of maps $h_1, \ldots, h_m : Y_d \to W$ such that

$$\sum_{j=0}^{d+1} (-1)^j \sum_{i=1}^{m} a_i \cdot (h_i \circ p_j) = 0$$

where $p_0, \ldots, p_{d+1} : Y_{d+1} \to Y_d$ are the structure maps of the simplicial objet $Y_\bullet$.

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1. an étale hyper-cover $Y_\bullet \to X$,
2. a (formal) linear combination $\sum_{i=1}^m a_i \cdot h_i$ of maps $h_1, \ldots, h_m : Y_d \to W$ such that

$$\sum_{j=0}^{d+1} (-1)^j \sum_{i=1}^{m} a_i \cdot (h_i \circ p_j) = 0$$

   where $p_0, \ldots, p_{d+1} : Y_{d+1} \to Y_d$ are the structure maps of the simplicial objet $Y_\bullet$.

An é-morphism is only defined up to an equivalence relation which we will ignore in this talk.

É-morphisms: more precisely (continued)

Remark (for those who know...)

É-morphisms of degree d from X to W form an Abelian group which can be identified with the étale cohomology group

$$H^d_{\text{ét}}(X, \mathbb{Z}_{\text{ét}}(W)),$$

where $\mathbb{Z}_{\text{ét}}(W)$ is the étale sheaf associated to the presheaf which sends an étale map $U \to X$ to the Abelian group of linear combinations of maps from U to W.

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Lemma

É-morphisms of degree 0 *from* X *to* W *are given by (linear combinations of) étale correspondences, i.e., by diagrams*

$$X \xleftarrow{e} X' \to W$$

where e *is an étale finite cover of* X.

Motivic morphisms

By the previous lemma, é-morphisms are not enough.
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Observation

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Definition

The n-th algebraic simplex is defined by

$$\Delta^n = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^n z_i = 1\}.$$

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We set $\Delta^n_X = \Delta^n \times X$.
Varying n, one gets a cosimplicial variety $\Delta^\bullet_X$ with a co-augmentation $X \to \Delta^\bullet_X$.

Motivic morphisms (continued)

Fact

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(Roughly speaking, this is a compatible collection of é-morphisms of degree $n + d$ from Δ^{n}_{X} to W.)

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Lemma

A motivic morphism of degree d from X to W induces a homomorphism $H^*(W) \to H^{*+d}(X)$ for any reasonable cohomology theory H^*.

Motives

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Definition

It is possible to compose motivic morphisms between algebraic varieties (as we compose maps). Thus, algebraic varieties and motivic morphisms form a category.

This category can be considerably enlarged by closing under homotopy limits and colimits. (A sort of a categorical completion.)

The resulting category is denoted by **DM**; this is the so-called Voevodsky's triangulated category of mixed motives.

Why does one care?

Slogan

Recall that when studying a mathematical object (or problem) it is often useful to look for symmetries; these usually have strong consequences and lead to simplifications, etc.

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Reformulation

The cohomological study of algebraic varieties is a chapter of representation theory! It is equivalent to the study of the representations of motivic (quasi-)symmetries.

This slogan can be made into conjectures which appear very hard nowadays. (E.g., interpreted in the right way, this slogan yields vast generalisations of the Hodge and Tate conjectures.)

Slogan (continued)

Here are three "qualitative statements" that are expected to hold.

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Note that the converses of these statements do hold and are relatively easy. (It is only a matter of formulating things correctly!) The main difficulty is to construct motivic morphisms (when we expect their presence). There is no general method for doing this!

Can one understand motivic morphisms?

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In practice this can be very hard, but there are some important special cases where things are easy.

1. If X and Y are smooth proper varieties, then motivic morphisms of degree 0 from X to Y are given by algebraic cycles in $X \times Y$ of dimension $\dim(X)$ up to rational equivalence, i.e., by the Chow group $\mathrm{CH}_{\dim(X)}(X \times Y)$.

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This is very nice, but rather useless... Indeed, it is well known that constructing interesting algebraic cycles is also very hard.

Some applications

The Milnor-Bloch-Kato conjecture

Fact

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Theorem (Voevodsky-Rost)

Let k be a field containing the n-th roots of unity (with n prime to the characteristic). Let G be the absolute Galois group of k.
Then $H^*(G, \mathbb{Z}/n\mathbb{Z})$ is isomorphic to the tensor $\mathbb{Z}/n\mathbb{Z}$-algebra on $k^\times/(k^\times)^n$ modulo the two sided ideal generated by the tenors $a \otimes (1-a)$ for $a \in k^\times \smallsetminus \{1\}$.

The work of Brown on multi-zetas

A more recent remarkable application of motivic ideas and techniques is due to F. Brown. He proved a conjecture of Hoffman on multiple zeta values:

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} \qquad \text{with} \quad n_i \geq 1,\ n_r \geq 2.$$

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Theorem (Brown)

Every multiple zeta value is a $\mathbb{Q}$-linear combination of $\zeta(n_1, \ldots, n_r)$ where $n_i \in \{2, 3\}$.

One of the main ingredients of the proof is a precise understanding of motivic morphisms between the so-called mixed Tate motives over $\mathbb{Q}$. (This understanding is a consequence of Borel's computation of algebraic K-theory of $\mathbb{Z}$.)

On the Kontsevich-Zagier conjecture

Recall that a period, in the sense of Kontsevich-Zagier, is a value of the pairing

$$\begin{array}{ccc} H_n^{\mathrm{Sing}}(X,Z) \otimes H^n_{\mathrm{DR}}(X,Z) & \to & \mathbb{C} \\ \gamma \otimes \omega & \mapsto & \int_\gamma \omega \end{array}$$

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where:

- X is an algebraic varieties defined over $\mathbb{Q}$,
- $Z \subset X$ is a closed sub-variety also defined over $\mathbb{Q}$,
- $H_n^{\mathrm{Sing}}(X,Z)$ is relative singular homology (a $\mathbb{Q}$-vector space),
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Kontsevich-Zagier Conjecture

$\mathbb{Q}$-linear relations between periods are produced by the usual rules of integration (i.e., change of variables and Stokes formula).

On the Kontsevich-Zagier conjecture (continued)

Using motivic techniques, it is possible to reformulate the conjecture of Kontsevich-Zagier more concretely as follows.

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Notation

Let $\mathcal{O}_{alg}(\bar{D}^{\infty})$ be the algebra of holomorphic functions on the unit polydisc in a finite (but unspecified) numbers of variables $t_1, \ldots, t_n, \ldots$, which are algebraic over the field of rational functions $\mathbb{Q}(t_1, \ldots, t_n, \ldots)$.

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The following conjecture is equivalent to the previous one.

Conjecture

The kernel of the integration on $[0,1]^\infty$:

$$\int_\square : \mathcal{O}_{alg}(\bar{D}^\infty) \to \mathbb{C}$$

is generated by elements of the form $\frac{\partial g}{\partial t_i} - g|_{t_i=1} + g|_{t_i=0}$.

On the Kontsevich-Zagier conjecture (continued)

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One can formulate a "geometric" version of this conjecture by replacing:

- $\mathbb{Q}$ by $\mathbb{C}(\varpi)$,
- $\mathcal{O}_{alg}(\bar{D}^{\infty})$ by the ring $\mathcal{O}^{\dagger}_{alg}(\bar{D}^{\infty})$ consisting of formal power series $\sum_{i>>-\infty} f_i(t_1, \ldots, t_n) \cdot \varpi^i$ with $f_i \in \mathcal{O}(\bar{D}^n)$ and which are algebraic over $\mathbb{C}(\varpi, t_1, \ldots, t_n)$.

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Interestingly, present motivic techniques are enough for proving the geometric version!

The end! Thank you